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CHARACTERIZATIONS OF REGULAR *po*-Γ-SEMIGROUPS

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Abstract

In this paper, we give two characterizations of regular *po*-Γ-semigroups in terms of left (resp., right, quasi-, bi-) ideals. The results generalize the results presented in [9].

1. Introduction and Preliminaries

Regular semigroups and their generalizations have been widely studied (see [4-9]). A semigroup S with a relation, denoted by $\leq$, is called

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a *partially ordered semigroup* (*po*-semigroup), if for $x, y, z \in S$; $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$. A *po*-semigroup S is said to be *regular*, if for any $a \in S$, there exists $x \in S$ such that $a \leq axa$. In 1999, the author [9] characterized regular *po*-semigroups by using some type of ideals. Indeed;

Theorem 1.1 [Theorem 1, p. 689]**.** *A po-semigroup S is regular, if and only if*

$$B \cap I \cap Q \subseteq (BIQ],$$

for all bi-ideal B, for all ideal I, and for all quasi-ideal Q of S.

Theorem 1.2 [Theorem 2, p. 690]**.** *A po-semigroup S is regular, if and only if*

$$R \cap Q \cap L \subseteq (RQL],$$

for all quasi-ideal Q, for all left ideal L, and for all right ideal R of S.

The purpose of this paper is to generalize Theorems 1.1 and 1.2 by using the concept of *po*-Γ-semigroup introduced by Sen and Saha [10].

Definition 1.3. Let S and Γ be two nonempty sets. Then S is called a Γ-*semigroup*, if there is a mapping $S \times \Gamma \times S \to S$, written as $(x, \gamma, y) \mapsto x\gamma y$, such that $(x\gamma y)\beta z = x\gamma(y\beta z)$ for all $x, y, z \in S$ and all $\gamma, \beta \in \Gamma$.

Let S be a semigroup and Γ be a nonempty set. For $x, y \in S$ and $\gamma \in \Gamma$, define $x\gamma y = xy$. Then S is a Γ-semigroup.

Let S be a Γ-semigroup. For $A, B \subseteq S$, let

$$A\Gamma B = \{a\gamma b \mid a \in A, b \in B, \gamma \in \Gamma\}.$$

For $x \in S$, let $A\Gamma x = A\Gamma\{x\}$ and $x\Gamma A = \{x\}\Gamma A$.

Definition 1.4. A Γ-semigroup S is called a partially ordered Γ-*semigroup* (*po*-Γ-semigroup), if there is a relation $\leq$ on S such that $x \leq y$ implies $x\gamma z \leq y\gamma z$ and $z\gamma x \leq z\gamma y$ for any $x, y, z \in S$ and all $\gamma \in \Gamma$.

Let S be a po-Γ-semigroup. For $A \subseteq S$, let

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

Let $A, B \subseteq S$. The following are well-known: (1) $A \subseteq (A]$, (2) If $A \subseteq B$, then $(A] \subseteq (B]$, and (3) $(A]\Gamma(B] \subseteq (A\Gamma B]$.

Definition 1.5. Let S be a po-Γ- semigroup. A nonempty subset A of S is called a *left* (resp., *right*) *ideal* of S, if

(i) $S\Gamma A \subseteq A$ (resp., $A\Gamma S \subseteq A$).

(ii) If $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

If A is both left and right ideals of S, then A is called an *ideal* of S.

Definition 1.6. Let S be a po-Γ- semigroup. A nonempty subset Q of S is called a *quasi-ideal* of S, if

(i) $(Q\Gamma S] \cap (S\Gamma Q] \subseteq Q$.

(ii) For $x \in Q$ and $y \in S$ such that $y \leq x$ implies $y \in Q$.

Definition 1.7. Let S be a po-Γ- semigroup. A nonempty subset B of S is called a *bi-ideal* of S, if

(i) $B\Gamma S\Gamma B \subseteq B$.

(ii) For $x \in B$ and $y \in S$ such that $y \leq x$ implies $y \in B$.

If A and B are left (resp., right, quasi-, bi-) ideal of a po-Γ- semigroup S, then $A = (A]$ and $(A \cap B] = (A] \cap (B]$.

Let S be a po-Γ- semigroup and $a \in S$. The right (resp., left) ideal of S generated by a, denoted by $R(a)$ (resp., $L(a)$) is of the form

$$R(a) = (a \cup a\Gamma S] \text{ (resp., } L(a) = (a \cup S\Gamma a]).$$

The ideal of S generated by a is

$$(a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S].$$

In [1-3], the authors proved that the quasi-(resp., bi-) ideal of S generated by a, denoted by $Q(a)$ (resp., $B(a)$) is of the form

$$Q(a) = (a \cup ((a\Gamma S] \cap (S\Gamma a]))] = ((a\Gamma S] \cap (S\Gamma a]] \cup (a]$$

(resp., $B(a) = (a \cup a\Gamma a \cup a\Gamma S\Gamma a]$).

Definition 1.8. A po-Γ-semigroup S is said to be *regular*, if $a \in (a\Gamma S\Gamma a]$ for all $a \in S$.

Equivalent definition. A po-Γ-semigroup S is said to be *regular*, if

$$a \leq a\alpha x\beta a,$$

for some $x \in S$ and for some $\alpha, \beta \in \Gamma$.

2. Regular po-Γ-Semigroups

Theorem 2.1. *A po-Γ-semigroup S is regular, if and only if*

$$B \cap I \cap Q \subseteq (B\Gamma I\Gamma Q],$$

for all bi-ideal B, for all ideal I, and for all quasi-ideal Q of S.

Proof. Assume that S is regular. Let $a \in B \cap I \cap Q$. By assumption, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$. Consider

$$\begin{aligned} a &\leq a\alpha x\beta a \\ &\leq (a\alpha x\beta a)\alpha x\beta(a\alpha x\beta a) \\ &= (a\alpha x\beta a)\alpha(x\beta a\alpha x)\beta a. \end{aligned}$$

Thus, $a \in (B\Gamma I\Gamma Q]$. Therefore, $B \cap I \cap Q \subseteq (B\Gamma I\Gamma Q]$.

Conversely, assume that

$$B \cap I \cap Q \subseteq (B\Gamma I\Gamma Q],$$

for all bi-ideal B, for all ideal I, and for all quasi-ideal Q of S. Let $a \in S$. Using $B(a)$, $I(a)$, and $Q(a)$ of S, we obtain

$$\begin{aligned} a &\in B(a) \cap I(a) \cap Q(a) \\ &\subseteq (B(a)\Gamma I(a)\Gamma Q(a)] \\ &\subseteq ((a \cup a\Gamma a \cup a\Gamma S\Gamma a]\Gamma S\Gamma(a \cup ((a\Gamma S] \cap (S\Gamma a])]] \\ &\subseteq (a\Gamma S\Gamma a]. \end{aligned}$$

Hence S is regular. □

For any po-Γ-semigroup, we note by the Definitions 1.5-1.7 that: Every left (resp., right) ideal is a quasi-ideal, and every qausi-ideal is a bi-ideal.

Corollary 2.2. *A* po-Γ-semigroup *S is regular, if and only if*

$$B \cap I \cap R \subseteq (B\Gamma I\Gamma R],$$

for all bi-ideal B, for all ideal I, and for all right ideal R of S.

Theorem 2.3. *A po-Γ-semigroup S is regular, if and only if*

$$R \cap Q \cap L \subseteq (R\Gamma Q\Gamma L],$$

for all quasi-ideal Q, for all left ideal L, and for all right ideal R of S.

Proof. Assume that S is regular. Let $a \in R \cap Q \cap L$. Then, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$. Consider

$$\begin{aligned} a &\leq a\alpha x\beta a \\ &\leq (a\alpha x\beta a)\alpha x\beta(a\alpha x\beta a) \\ &= (a\alpha x)\beta a\alpha(x\beta a\alpha x\beta a). \end{aligned}$$

Thus, $a \in (R\Gamma Q\Gamma L]$. Hence, $R \cap Q \cap L \subseteq (R\Gamma Q\Gamma L]$.

Conversely, assume that

$$R \cap Q \cap L \subseteq (R\Gamma Q\Gamma L],$$

for all quasi-ideal Q, for all left ideal L, and for all right ideal R of S. To show that S is regular, let $a \in S$. Using $Q(a)$, $L(a)$, and $R(a)$ of S by assumption, we have

$$\begin{aligned} a &\in R(a) \cap Q(a) \cap L(a) \\ &\subseteq (R(a)\Gamma Q(a)\Gamma L(a)] \\ &\subseteq (R(a)\Gamma S\Gamma L(a)] \\ &\subseteq (R(a)\Gamma L(a)] \\ &\subseteq ((a \cup a\Gamma S]\Gamma(a \cup S\Gamma a]] \\ &\subseteq (a\Gamma a \cup a\Gamma S\Gamma a]. \end{aligned}$$

There are two cases to consider: If $a \leq a\gamma a$ for some $\gamma \in \Gamma$, then

$$a \leq a\gamma a \leq a\gamma(a\gamma a)\gamma a \in a\Gamma S\Gamma a.$$

If $a \leq a\gamma x\beta a$ for some $x \in S$ and for some $\alpha, \beta \in \Gamma$, then $a \in a\Gamma S\Gamma a$. Both of the cases we obtain $a \in (a\Gamma S\Gamma a]$. □

Corollary 2.4. *A po-*Γ*-semigroup S is regular, if and only if*

$$R \cap L \subseteq (R\Gamma L],$$

for all left ideal L and for all right ideal R of S.

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